Computational Complexity of Barwise’s Sentence and Similar Natural Language Constructions

Dariusz Kalociński
Michał Tomasz Godziszewski
University of Warsaw, Poland

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Computational Semantics of Natural Language

Meaning as algorithm:

- Ability of understanding sentences: capacity of recognizing their truth-values.
- Treat the meaning as a computational procedure of finding extension of some expression in a model, i.e. as an algorithm for verifying the truth-value of a sentence.
- Complexity Sentence verification: how difficult are certain natural language constructions?
- Better understanding of linguistic competence - valuable for Cognitive Science
- Classification of semantic constructions
Barwise’s sentence (BS)

The richer the country the more powerful are some of its officials.

and other variations of “the ... the ...” construction.

Question 1
What are reasonable logical forms for BS and similar constructions?

Question 2
What is the computational complexity of recognizing the truth value of BS and similar NL constructions in finite models?

Results
Some of them are P, some NP-complete...
Logical form for BS

Barwise’s sentence (BS)

*The richer the country the more powerful are some of its officials.*

Barwise’s claim
This sentence expresses that there is a homomorphism of one partial order into the other.

$$\exists f \forall x, y \in C [x > y \Rightarrow (O(f(x), x) \land O(f(y), y) \land f(x) \succ f(y))]$$
Strict Partial Orders Homomorphism (SPOH)

Definition (homomorphism between strict posets)
Let \(A\) and \(B\) be strict posets and \(f : A \rightarrow B\). \(f\) is a homomorphism of \(A\) into \(B\), in symbols \(f : A \hookrightarrow B\), iff

\[
(\forall x, y \in A) (x <_A y \Rightarrow f(x) <_B f(y)).
\]

We say \(A\) is homomorphic to \(B\), in symbols \(A \hookrightarrow B\), iff there is an embedding \(f\) of \(A\) into \(B\).

Definition (SPOH)

Input: strict partial orders \(A = (A, <_A)\), \(B = (B, <_B)\)
Question: is there a homomorphism from \(A\) to \(B\)?

Theorem

\(SPOH\) is \(PTIME\).

Corollary

Recognizing the truth value of \(BS\) in finite models is \(PTIME\)-computable.
Definition
Let $\mathcal{A} = (A, <_A)$ be a finite strict partial order. The height of $\mathcal{A}$, denoted by $h(\mathcal{A})$, is the number of vertices in the longest chain in $\mathcal{A}$.

Lemmas
1. $h$ is PTIME,
2. for any strict partial orders $\mathcal{A}, \mathcal{B}$:

$$\mathcal{A} \hookrightarrow \mathcal{B} \iff h(\mathcal{B}) \geq h(\mathcal{A})$$

Theorem
$SPOH$ is PTIME.
Computing $h$ in PTIME

Approach 1 (Sedgewick)

_input_: a strict poset $\mathcal{X}$

_output_: the height of $\mathcal{X}$

1. $\mathcal{X}' =$ weighted strict poset $\mathcal{X}$, where each vertex has the weight $-1$.

2. Run the Ford-Bellman’s algorithm on $\mathcal{X}'$ (observe that the shortest path in $\mathcal{X}'$ is the longest path in $\mathcal{X}$).

3. Return the number of vertices of the shortest path in $\mathcal{X}'$.

Approach 2

A modification of BFS algorithm?
Homomorphism in terms of posets’ height

Definition (source vertex)
A source vertex of a partial order is any vertex that has no ingoing edges.

Definition (poset with artificial source)
Let $\mathcal{X}$ be a strict poset and $s \notin \mathcal{X}$. We say a strict poset $\mathcal{Y}$ is $\mathcal{X}$ with artificial source $s$ iff $\mathcal{Y} = \mathcal{X} \cup \{s\}$ and $\prec_{\mathcal{Y}}$ is the least strict partial order containing $\prec_{\mathcal{X}}$ and having edges from $s$ to any source vertex of $\mathcal{X}$.

Definition (levels of partial order)
Let $\mathcal{X} = (\mathcal{X}, \prec_{\mathcal{X}})$ be a partial order and $\mathcal{X}'$ be $\mathcal{X}$ with artificial source $s$. Let $n \in \{1, 2, \ldots, h(\mathcal{X})\}$. We define the $n^{th}$ level of $\mathcal{X}$, denoted by $X_n$. For all $x \in X$:

$$x \in X_n \iff \text{the longest chain from } s \text{ to } x \text{ in } \mathcal{X}' \text{ has length } n.$$
Figure 1:  Homomorphism $g$ from $A$ to $B$, where $h(B) \geq h(A)$ and $b_1b_2 \ldots b_{h(B)-1}$, $b_{h(B)}$ is the longest chain in $B$. 
Similar NL constructions

The smarter the student the better are some of her individual presentations AND the better are these presentations the smarter are students who performed them.

Claim
This sentence seems to express that there is an embedding from one partial order to the other.

\[ \exists f \forall x, y \in S [ x \succ y \iff (P(f(x), x) \land P(f(y), y) \land f(x) \succ f(y))] \]
Strict Partial Orders Embedding (SPOE)

**Definition (strict posets embedding)**
Let $A$, $B$ be strict posets and $f : A \rightarrow B$. $f$ is an embedding of $A$ into $B$, in symbols $f : A \llcorner e \rightarrow B$, iff $f$ is injective and

$$(\forall x, y \in A) (x <_A y \iff f(x) <_B f(y)).$$

We say $A$ embeds in $B$, in symbols $A \llcorner e \rightarrow B$, iff there is an embedding $f$ of $A$ into $B$.

**Definition (SPOE)**

**Input**: strict partial orders $A = (A, <_A)$, $B = (B, <_B)$

**Question**: is there an embedding $f$ from $A$ to $B$?

**Theorem**

*SPOE is NP-complete.*
SPOE is NP-complete

Idea of proof.

- SPOE is in NP - easy.
- Polynomial reduction of 3SAT to SPOE:

$$3\text{CNF} \ni \varphi \mapsto (A_\varphi, B_\varphi)$$

$$3\text{CNF} \ni \varphi := (a_1 \lor b_1 \lor c_1) \land (a_2 \lor b_2 \lor c_2) \land \ldots \land (a_k \lor b_k \lor c_k)$$
Idea of proof cd.

\[ \varphi := (a_1 \lor b_1 \lor c_1) \land (a_2 \lor b_2 \lor c_2) \land \ldots \land (a_k \lor b_k \lor c_k) \]

Figure 2: \( \varphi \mapsto A_\varphi \)
Idea of proof cd.

\[ \varphi := (a_1 \lor b_1 \lor c_1) \land (a_2 \lor b_2 \lor c_2) \land \ldots \land (a_k \lor b_k \lor c_k) \]

Figure 3: \( \varphi \mapsto B_\varphi \)
\[ \varphi := (a_1 \lor b_1 \lor c_1) \land (a_2 \lor b_2 \lor c_2) \land \ldots \land (a_k \lor b_k \lor c_k) \]

\[ \varphi \in 3\text{SAT} \]

iff
What about recognizing the truth value?

The smarter the student the better are some of her individual presentations AND the better are these presentations the smarter are students who performed them.

\[ \exists f \forall x, y \in A [x > y \Leftrightarrow (P(f(x), x) \land P(f(y), y) \land f(x) > f(y))] \]

Not exactly SPOE - given a student, \( f \) cannot choose any presentation, but only student’s individual presentation.

**Definition (Strict Partial Orders Embedding in Partition, SPOEP)**

**Input:** strict partial orders \( \mathcal{A}, \mathcal{B} \), a partition \( \{B_a\}_{a \in A} \) such that \( \bigcup_{a \in A} B_a \subseteq B \).

**Question**: is there an embedding \( f \) of \( \mathcal{A} \) into \( \mathcal{B} \) s.t. \( f(a) \in B_a \), for every \( a \in A \)?
Recognizing the truth value

**Theorem**

*SPOEP is NP-complete.*

**Proof.**

Reduction of SPOE to SPOEP.

**Theorem (imprecise formulation)**

*Recognizing the truth value of*

\[ \exists f \forall x, y \in S \ [x > y \iff (P(f(x), x) \land P(f(y), y) \land f(x) \succ f(y))] \]

*in finite models, where interpretation of P induces a partition, is NP-complete.*

**Proof.**

Reduction of SPOEP to this problem.
Similar NL constructions

The smarter the student the better are some of her individual presentations.

Claim
This sentence seems to express that there is an 1-1 homomorphism from one partial order to the other.

\[ \exists f \forall x, y \in S \left[ x > y \Rightarrow (P(f(x), x) \land P(f(y), y) \land f(x) \succ f(y)) \right] \]

Definition (Strict Posets Injective Homomorphism, SPOIH)

Input: strict partial orders \( \mathcal{A} = (A, <_A) \), \( \mathcal{B} = (B, <_B) \)

Question: is there a 1-1 homomorphism from \( \mathcal{A} \) to \( \mathcal{B} \)?

Theorem

\( SPOIH \) is NP-complete.
SPOIH is NP-complete - idea of proof

\[(a_1 \lor b_1 \lor c_1) \land (a_2 \lor b_2 \lor c_2) \land (a_3 \lor b_3 \lor c_3) \land (a_4 \lor b_4 \lor c_4)\]
The smarter the student the better are some of her individual presentations.

\[ \exists f \forall x, y \in S [x > y \Rightarrow (P(f(x), x) \land P(f(y), y) \land f(x) > f(y))] \]

Not exactly SPOIH - given a student, \( f \) cannot choose any presentation, but only student’s individual presentation.

**Definition (Strict Posets Homomorphism in Partition, SPOIHP)**

**Input:** strict partial orders \( A, B \), a partition \( \{ B_a \}_{a \in A} \) such that \( \bigcup_{a \in A} B_a \subseteq B \).

**Question:** is there a 1-1 homomorphism \( g \) of \( A \) into \( B \) s.t. \( g(a) \in B_a \), for every \( a \in A \)?
Recognizing the truth value

Hypothesis 1
SPOIHP is NP-complete.

Try
Reduction of SPOIH to SPOIHP.

Hypothesis 2 - imprecise formulation
Recognizing the truth value of
\[ \exists f \forall x, y \in S \left[ x > y \iff (P(f(x), x) \land P(f(y), y) \land f(x) \succ f(y)) \right], \]
in finite models, where the interpretation of \( P \) induces a partition, is NP-complete.

Try
Reduction of SPOIHP to this problem.
Thank you for your attention!
K. Jon Barwise.
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