Computational Complexity of Barwise's Sentence and Similar Natural Language Constructions

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Computational Semantics of Natural Language

Meaning as algorithm:

- Ability of understanding sentences: capacity of recognizing their truth-values.
- Treat the meaning as a computational procedure of finding extension of some expression in a model, i.e. as an algorithm for verifying the truth-value of a sentence.
- Complexity Sentence verification: how difficult are certain natural language constructions?
- Better understanding of linguistic competence - valuable for Cognitive Science
- Classification of semantic constructions

Outline

Barwise's sentence (BS)

The richer the country the more powerful are some of its officials.

and other variations of "the ... the ..." construction.

Question 1

What are reasonable logical forms for BS and similar constructions?

Question 2

What is the computational complexity of recognizing the truth value of BS and similar NL constructions in finite models?

Results

Some of them are P, some NP-complete...

Logical form for BS

Barwise's sentence (BS)

The richer the country the more powerful are some of its officials.

Barwise's claim

This sentence expresses that there is a homomorphism of one partial order into the other.

$$\exists f \, \forall x, y \in C[x > y \Rightarrow (O(f(x), x) \wedge O(f(y), y) \wedge f(x) \succ f(y))]$$

Strict Partial Orders Homomorphism (SPOH)

Definition (homomorphism between strict posets)

Let $\mathcal{A}$, $\mathcal{B}$ be strict posets and $f : A \to B$. f is a homomorphism of $\mathcal{A}$ into $\mathcal{B}$, in symbols $f : \mathcal{A} \hookrightarrow \mathcal{B}$, iff

$$(\forall\, x, y \in A)\,(x <_A y \Rightarrow f(x) <_B f(y)).$$

We say $\mathcal{A}$ is homomorphic to $\mathcal{B}$, in symbols $\mathcal{A} \hookrightarrow \mathcal{B}$, iff there is an embedding f of $\mathcal{A}$ into $\mathcal{B}$.

Definition (SPOH)

Input: strict partial orders $\mathcal{A} = (A, <_A)$, $\mathcal{B} = (B, <_B)$
Question: is there a homomorphism from $\mathcal{A}$ to $\mathcal{B}$?

Theorem

SPOH is PTIME.

Corollary

Recognizing the truth value of BS in finite models is PTIME-computable.

SPOH is PTIME - idea of proof

Definition

Let $\mathcal{A} = (A, <_A)$ be a finite strict partial order. The height of $\mathcal{A}$, denoted by $h(\mathcal{A})$, is the number of vertices in the longest chain in $\mathcal{A}$.

Lemmas

1. h is PTIME,
2. for any strict partial orders $\mathcal{A}$, $\mathcal{B}$:

$$\mathcal{A} \hookrightarrow \mathcal{B} \Leftrightarrow h(\mathcal{B}) \geq h(\mathcal{A})$$

Theorem

SPOH is PTIME.

Computing h in PTIME

Approach 1 (Sedgewick)

input: a strict poset $\mathcal{X}$
output: the height of $\mathcal{X}$

1. $\mathcal{X}' =$ weighted strict poset $\mathcal{X}$, where each vertex has the weight -1.
2. Run the Ford-Bellman's algorithm on $\mathcal{X}'$ (observe that the shortest path in $\mathcal{X}'$ is the longest path in $\mathcal{X}$).
3. Return the number of vertices of the shortest path in $\mathcal{X}'$.

Approach 2

A modification of BFS algorithm?

Homomorphism in terms of posets' height

Definition (source vertex)

A source vertex of a partial order is any vertex that has no ingoing edges.

Definition (poset with artificial source)

Let $\mathcal{X}$ be a strict poset and $s \notin X$. We say a strict poset $\mathcal{Y}$ is $\mathcal{X}$ with artificial source s iff $Y = X \cup \{s\}$ and $<_Y$ is the least strict partial order containing $<_X$ and having edges from s to any source vertex of $\mathcal{X}$.

Definition (levels of partial order)

Let $\mathcal{X} = (X, <_X)$ be a partial order and $\mathcal{X}'$ be $\mathcal{X}$ with artificial source s. Let $n \in \{1, 2, \ldots, h(\mathcal{X})\}$. We define the n^{th} level of $\mathcal{X}$, denoted by X_n. For all $x \in X$:

$$x \in X_n \Leftrightarrow \text{the longest chain from } s \text{ to } x \text{ in } \mathcal{X}' \text{ has length } n.$$

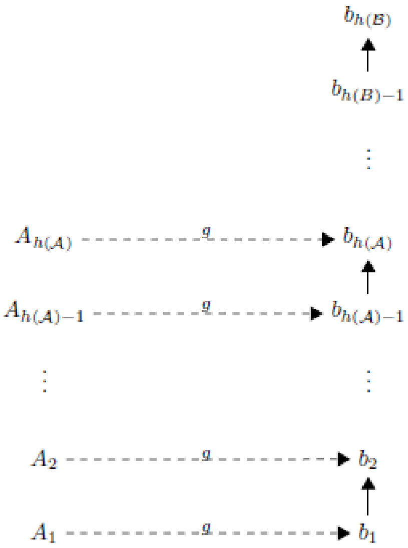


Figure 1: Homomorphism g from $\mathcal{A}$ to $\mathcal{B}$, where $h(\mathcal{B}) \geq h(\mathcal{A})$ and $b_1 b_2 \ldots b_{h(\mathcal{B})-1}, b_{h(\mathcal{B})}$ is the longest chain in $\mathcal{B}$.

Similar NL constructions

The smarter the student the better are some of her individual presentations <u>*AND*</u> *the better are these presentations the smarter are students who performed them.*

Claim

This sentence seems to express that there is an embedding from one partial order to the other.

$$\exists f\, \forall x, y \in S[x > y \Leftrightarrow (P(f(x), x) \wedge P(f(y), y) \wedge f(x) \succ f(y))]$$

Strict Partial Orders Embedding (SPOE)

Definition (strict posets embedding)

Let $\mathcal{A}$, $\mathcal{B}$ be strict posets and $f : A \to B$. f is an embedding of $\mathcal{A}$ into $\mathcal{B}$, in symbols $f : \mathcal{A} \overset{e}{\hookrightarrow} \mathcal{B}$, iff f is injective and

$$(\forall x, y \in A)\,(x <_A y \Leftrightarrow f(x) <_B f(y)).$$

We say $\mathcal{A}$ embeds in $\mathcal{B}$, in symbols $\mathcal{A} \overset{e}{\hookrightarrow} \mathcal{B}$, iff there is an embedding f of $\mathcal{A}$ into $\mathcal{B}$.

Definition (SPOE)

Input: strict partial orders $\mathcal{A} = (A, <_A)$, $\mathcal{B} = (B, <_B)$
Question: is there an embedding f from $\mathcal{A}$ to $\mathcal{B}$?

Theorem

SPOE is NP-complete.

SPOE is NP-complete

Idea of proof.

- SPOE is in NP - easy.
- Polynomial reduction of 3SAT to SPOE:

$$\mathrm{3CNF} \ni \varphi \mapsto (\mathcal{A}_\varphi, \mathcal{B}_\varphi)$$

$3\mathrm{CNF} \ni \varphi := (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \ldots \wedge (a_k \vee b_k \vee c_k)$

Idea of proof cd.

$$\varphi := (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \ldots \wedge (a_k \vee b_k \vee c_k)$$

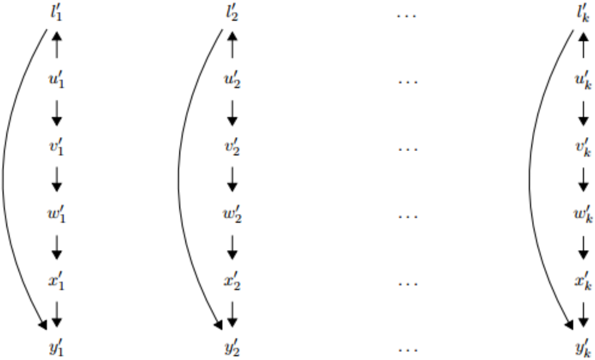


Figure 2: $\varphi \mapsto \mathcal{A}_\varphi$

Idea of proof cd.

$$\varphi := (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \ldots \wedge (a_k \vee b_k \vee c_k)$$

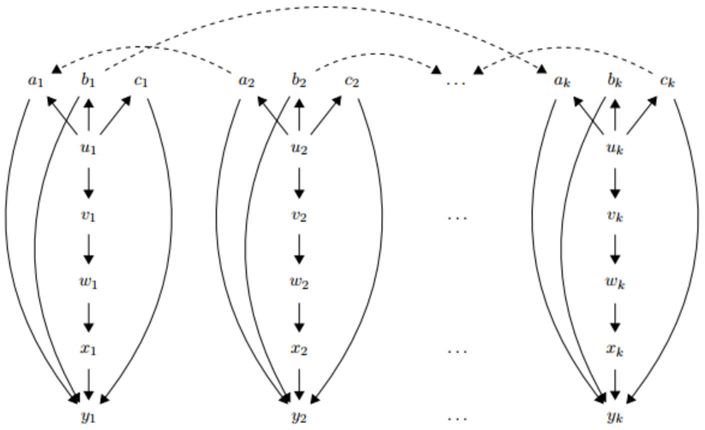
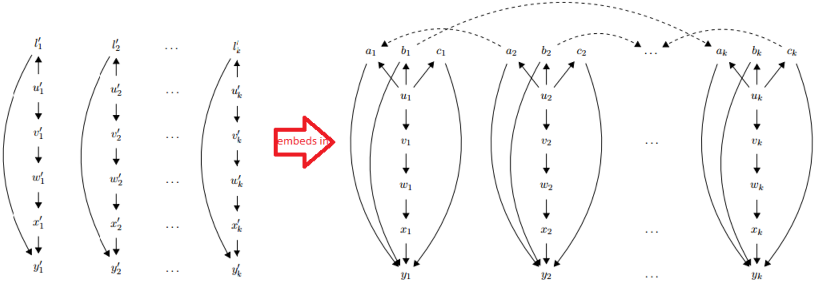


Figure 3: $\varphi \mapsto \mathcal{B}_\varphi$

$$\varphi := (a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \ldots \wedge (a_k \vee b_k \vee c_k)$$

$$\varphi \in \mathsf{3SAT}$$

iff



What about recognizing the truth value?

The smarter the student the better are some of her individual presentations AND the better are these presentations the smarter are students who performed them.

$$\exists f\, \forall x, y \in A\, [x > y \Leftrightarrow (P(f(x), x) \wedge P(f(y), y) \wedge f(x) \succ f(y))]$$

Not exactly SPOE - given a student, f cannot choose any presentation, but only student's individual presentation.

Definition (Strict Partial Orders Embedding in Partition, SPOEP)

Input: strict partial orders $\mathcal{A}$, $\mathcal{B}$, a partition $\{B_a\}_{a \in A}$ such that $\bigcup_{a \in A} B_a \subseteq B$.
Question: is there an embedding f of $\mathcal{A}$ into $\mathcal{B}$ s.t. $f(a) \in B_a$, for every $a \in A$?

Recognizing the truth value

Theorem

SPOEP is NP-complete.

Proof.

Reduction of SPOE to SPOEP. $\square$

Theorem (imprecise formulation)

Recognizing the truth value of
$\exists\, f\, \forall x, y \in S\, [x > y \Leftrightarrow (P(f(x), x) \wedge P(f(y), y) \wedge f(x) \succ f(y))]$
in finite models, where interpretation of P induces a partition, is NP-complete.

Proof.

Reduction of SPOEP to this problem. $\square$

Similar NL constructions

The smarter the student the better are some of her individual presentations.

Claim

This sentence seems to express that there is an 1-1 homomorphism from one partial order to the other.

$$\exists f \, \forall x, y \in S[x > y \Rightarrow (P(f(x), x) \wedge P(f(y), y) \wedge f(x) \succ f(y))]$$

Definition (Strict Posets Injective Homomorphism, SPOIH)

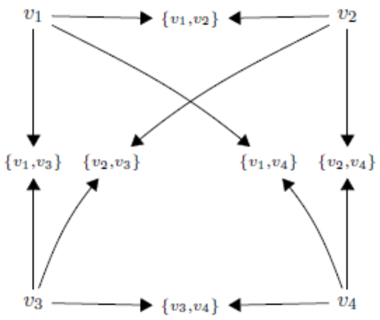
Input: strict partial orders $\mathcal{A} = (A, <_A)$, $\mathcal{B} = (B, <_B)$
Question: is there a 1-1 homomorphism from $\mathcal{A}$ to $\mathcal{B}$?

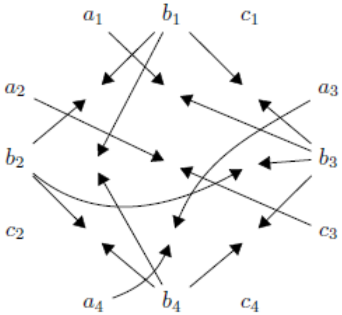
Theorem

SPOIH is NP-complete.

SPOIH is NP-complete - idea of proof

$$(a_1 \lor b_1 \lor c_1) \land (a_2 \lor b_2 \lor c_2) \land (a_3 \lor b_3 \lor c_3) \land (a_4 \lor b_4 \lor c_4)$$





The smarter the student the better are some of her individual presentations.

$$\exists\, f\, \forall x, y \in S[x > y \Rightarrow (P(f(x), x) \wedge P(f(y), y) \wedge f(x) \succ f(y))]$$

Not exactly SPOIH - given a student, f cannot choose any presentation, but only student's individual presentation.

Definition (Strict Posets Homomorphism in Partition, SPOIHP)

Input: strict partial orders $\mathcal{A}$, $\mathcal{B}$, a partition $\{B_a\}_{a \in A}$ such that $\bigcup_{a \in A} B_a \subseteq B$.
Question: is there a 1-1 homomorphism g of $\mathcal{A}$ into $\mathcal{B}$ s.t. $g(a) \in B_a$, for every $a \in A$?

Recognizing the truth value

Hypothesis 1

SPOIHP is NP-complete.

Try

Reduction of SPOIH to SPOIHP.

Hypothesis 2 - imprecise formulation

Recognizing the truth value of
$\exists\, f\, \forall x, y \in S\, [x > y \Leftrightarrow (P(f(x), x) \wedge P(f(y), y) \wedge f(x) \succ f(y))]$,
in finite models, where the interpretation of P induces a partition, is NP-complete.

Try

Reduction of SPOIHP to this problem.

Thank you for your attention!

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