

# Computational Complexity of the Barwise's Sentence and Similar Natural Language Constructions

We consider examples of a natural language construction such as "The richer the country, the more powerful are some of its officials" (see [Bar78]). We argue that the adequate logical form of sentences such as the one presented by Barwise expresses the fact that one strict partial order is homomorphic to the other. We show that if this interpretation is correct, then recognizing the truth value of such sentences in finite models is PTIME-computable. Besides, we consider some other examples of the construction in question, such as "Those professors, whose some students earn higher salaries, are exactly the wiser ones" and "The wiser the professor the more wealthy are some of her PhD students". Their postulated logical forms express the fact that one strict partial order is, respectively, embedded into and one-one homomorphic to the other (we may need some easy assumptions guaranteeing injectivity). Both interpretations imply that the problem of recognizing the truth value of such sentences in finite models is NP-complete.

Specifically, we define the following computational problems:

**Definition 1.**

$$\text{SPOH} = \{(\mathcal{A}, \mathcal{B}) : \mathcal{A} \text{ and } \mathcal{B} \text{ are strict posets such that } \mathcal{A} \text{ is homomorphic to } \mathcal{B}\} \quad (1a)$$

$$\text{SPOIH} = \{(\mathcal{A}, \mathcal{B}) : \mathcal{A} \text{ and } \mathcal{B} \text{ are strict posets such that } \mathcal{A} \text{ is 1-1 homomorphic to } \mathcal{B}\} \quad (1b)$$

$$\text{SPOE} = \{(\mathcal{A}, \mathcal{B}) : \mathcal{A} \text{ and } \mathcal{B} \text{ are posets such that } \mathcal{A} \text{ embeds into } \mathcal{B}\} \quad (1c)$$

and we prove the following:

**Theorem 1.** SPOE is NP-complete.

We demonstrate that 3SAT is polynomially reducible to SPOE. We describe a polynomial algorithm that takes an arbitrary formula  $\varphi$  in 3CNF as input and returns an ordered pair of strict partial orders  $\mathcal{A}_\varphi, \mathcal{B}_\varphi$  satisfying the following condition:

$$\forall \varphi \in 3\text{CNF} (\varphi \in 3\text{SAT} \Leftrightarrow \mathcal{A}_\varphi \text{ embeds into } \mathcal{B}_\varphi). \quad (2)$$

**Theorem 2.** SPOIH is NP-complete.

We show that that 3SAT is polynomially reducible to SPOIH, i.e. for any 3CNF formula  $\varphi$  there is a pair  $\mathcal{A}, \mathcal{B}$  of strict posets such that  $\varphi$  is satisfiable if and only if there is an injective homomorphism  $h : \mathcal{A} \rightarrow \mathcal{B}$  and the construction of  $\mathcal{A}$  and  $\mathcal{B}$  from  $\varphi$  is polynomial.

**Theorem 3.** SPOH is in PTIME.

**Definition 2.** Let  $\mathcal{A} = (A, <_A)$  be a finite strict partial order. The height of  $\mathcal{A}$ , denoted by  $h(\mathcal{A})$ , is the number of vertices in the longest chain in  $\mathcal{A}$ .

**Lemat 1.**  $h$  is PTIME.

The algorithm takes a strict poset  $\mathcal{X}$  as input. Let  $\mathcal{X}'$  become the weighted strict poset  $\mathcal{X}$ , where each vertex has the weight  $-1$ . Run the Ford-Bellman's algorithm on  $\mathcal{X}'$ . Return the number of vertices of the shortest path in  $\mathcal{X}'$ .

**Lemat 2.** For any strict partial orders  $\mathcal{A}, \mathcal{B}$ :

$$\mathcal{A} \hookrightarrow \mathcal{B} \Leftrightarrow h(\mathcal{A}) \leq h(\mathcal{B})$$

The last theorem easily follows from the two lemmas.

We conclude with the methodological discussion on the algorithmic approaches to the philosophical theory of meaning and we discuss possible ways of establishing the logical form of problematic natural language constructions.

## References

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